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by A. V. Gurevich

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THE THEORY OF CROSS-MODULATION OF RADIOWAVES

A. V. Gurevich

ABSTRACT 24627

The percentage of cross-modulation is examined as a function of the radiated power of the interfering station for the case of a strong interfering wave. The topics of cross-modulation resonance near the gyrofrequency and the influence of the electron velocity distribution on the phase of the cross-modulation are also considered.

AUTHOR

Introduction

Perturbations caused by powerful radiowaves in the ionosphere affect waves propagated in the perturbed region. Specifically, in the case when the perturbing wave is amplitude modulated with a low frequency Ω, then the perturbations which it causes in the plasma are modulated, and, hence, so are other waves which have passed through the perturbed region. This phenomenon is known as cross-modulation, or the Luxemburg effect.

The theory of the Luxemburg effect, first formulated by Bailey and Martyn in Ref. 1 (see Refs. 2-7 also), predicts that the depth μ' and the phase φ' of the cross-modulation will exhibit a definite dependence on the modulation frequency Ω and also on the carrier frequency ω_1 and the power P of the interfering wave.

The dependence of the depth and phase of cross-modulation on the modulation frequency and the percentage of modulation of the perturbing wave was investigated experimentally in Refs. 8-10; experimentation has, in general, agreed well with the conclusions of theory. The percentage of cross-modulation as a function of the power of the interfering station was examined in Refs. 8, 5, 11; it was shown that μ' is proportional to P right up to P = 520 kw, the maximum radiated power in use. A comparison of these results with theory (Refs. 4, 5) led to the conclusion that a wave of radiated power up to 520 kw is weak, i.e., it changes the effective electron temperature in the ionosphere only to an insignificant degree (less than 10 percent). This conclusion, however, does not agree well with the results of calculation, nor with the results of other experiments. For example, in Ref. 8, the percentage of cross-modulation reached 0.2 at substantially less power, $P \sim 150$ kw, which is possible only if the effective electron temperature in the ionosphere changes by 20-50 percent.[1]

Experimental research into the dependence of the percentage of cross-modulation on the frequency of the perturbing wave ω_1 has been dedicated, mainly, to the question of the resonant increase in the percentage of cross-modulation which theory predicts near the gyrofrequency (at $\omega_1 \approx \omega_H = eH/mc$). However, the results of this research are contradictory: the indicated resonance has been observed in Refs. 15, 16, whereas in Refs. 5, 11, 17 no intensification of the cross-modulation at all was recorded in the vicinity of the gyrofrequency; part of the work on this problem yielded no definite results (Refs. 9, 18). We note also that, as pointed out in Ref. 7, Section 110, experimental results do not support yet another conclusion of theory which states that for perturbing waves of very low frequency ($\omega_1 \ll \omega_H$), the percentage of cross-modulation must decrease, proportional to ω_1^{-2}, with an increase in ω_1.

Hence, we can conclude that the existing theory does not agree well with experiment on the question of the dependence of the percentage of cross-modulation on the frequency of the interfering wave. It is not hard to see one possible reason for this lack of agreement. As a matter of fact, in deriving the final equations, the theory assumes, for simplicity, that the percentage of cross-modulation is proportional to the perturbation caused by the interfering wave at a certain "average" point in the region of interaction. However, under such an assumption it is impossible to obtain the correct dependence of the percentage of cross-modulation on the frequency of the disturbing wave ω_1. This is because,

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[1]In this regard, Huxley in Ref. 12 developed a new "alternative" theory, which led to a linear dependence of μ' on P for a strong interfering wave, while the dependence of μ' on P in a weak field turns out to be markedly nonlinear. Since the latter conclusion is in direct contradiction with experiment (Refs. 8, 13), Huxley later practically denied the proposed theory (Ref. 14). We might also add that the conclusion about the linear dependence of μ' on P in the "alternative" theory is not true for the case of a strong interfering wave, since in working out his theory Huxley did not make allowance for the self-interaction of the interfering wave (see below).

with a change in the frequency ω_1, not only does the perturbation caused by the wave at an "average" point in the region of interaction substantially change, but also the effective dimensions of the region itself change. The latter circumstance must be taken into account in the theory. It is possible to accomplish this most completely by summing (integrating) the influence of the perturbations on the wave experiencing cross-modulation over the whole region of interaction.[1]

Further, as was noted above, the problem of the cross-modulation for the case of a very powerful ("strong", according to Ref. 19) interfering wave, which substantially changes the effective temperature of the electrons in the plasma, is also important for experiment. However, the existing theory given by Refs. 2, 6 is not sufficiently accurate in this case for the following reasons. In the first place, it does not allow for the "self-interaction" of the interfering wave, whereas an allowance for the self-interaction of a strong interfering wave is necessary in the theory of cross-modulations. As a matter of fact, as is shown in Refs. 19, 20, the amplitude of the field voltage, the depth and phase of the modulation of a strong wave in plasma are substantially dependent on its power and on the frequency of the wave; all this cannot help but be evidenced in the depth and phase of cross-modulation. Secondly, the perturbations caused in the plasma by a strong wave are not linearly dependent on its power, and this circumstance demands an "integral" examination (as indicated above) in the computation of the dependence of the percentage of cross-modulation on the radiated power of the disturbing station.

Such an integral examination is carried out below in Sections 1 and 2 of the present work. In Section 1, expressions for the cross-modulation depth μ' are obtained for the case of a strong perturbing wave, considering its self-interaction; the dependence of μ' on the radiated power of the interfering station is examined. In Section 2, the topic of cross-modulation resonance in the vicinity of the gyrofrequency is examined. It is shown that the results of the theory agree well with the experimental data available at the present time.

In Section 3, the phase of the cross-modulation as a function of the modulation frequency of the interfering wave is found, taking into

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[1]Such an "integral" examination was carried out by Shaw in Ref. 5 (see also Refs. 4, 13). However, it was not sufficiently well developed, and led to incorrect conclusions: for example, the author's conclusion that there is no resonance of the cross-modulation at the gyrofrequency is false.

account the electron velocity distribution in the ionosphere; a deviation is obtained from the well-known results of theory which does not take into account the electron velocity distribution (Ref. 1). Apparently, the deviation can be checked experimentally.

1. Cross-Modulation for the Case of a Strong Perturbing Wave

Let a strong wave approach the boundary of the plasma

$$E_1 = E_0(1 + \mu_0 \cos \Omega t) \cos \omega_1 t. \tag{1}$$

Under the influence of its electric field, as the wave propagates in the plasma, both constant and time-varying (with frequency Ω) perturbations of the effective electron temperature arise. Due to this the absorption of the second wave,

$$E_2 = E_{20} \cos \omega_2 t,$$

propagating in the perturbed region changes. It is, therefore, modulated with the same modulation frequency Ω.

First, we will examine the case of low frequency modulation $\Omega \ll \delta\nu$, where ν is the effective collision frequency of an electron (Ref. 7, Section 61; Ref. 21), and δ is the mean fraction of energy transferred by the electron to a molecule in one collision; for the ionosphere $\delta \approx 2 \cdot 10^{-3}$. In this case, it is not hard to obtain the expression for the percentage of cross-modulation (taking the self-interaction into account) by making the following simple considerations. We will examine a perturbed layer in the plasma of thickness dz (from z to z + dz). The percentage of cross-modulation of the wave E_2, passing through this layer, increases by $d\mu'$; from the definition of the percentage of cross-modulation it follows that

$$d\mu' = d\left[(E_{2max} - E_{2min})/(E_{2max} + E_{2min})\right], \tag{2}$$

where E_{2max} and E_{2min} are respectively the maximum and minimum values which are taken on by the time varying wave field E_2 at point z. Assume that the percentage of modulation of the interfering wave at point z is not large ($\mu \ll 1$). Then recalling that $E_{2max} = E_2(E_1 \mp \mu E_1)$ and

$E_{2min} = E_2(E_1 \pm \mu E_1)$ (the upper sign is chosen for $\omega_2 > \nu$, the lower $\omega_2 < \nu$) and expanding $E_2(\mu)$ in equation (2) in a power series in μ, we find that

$$d\mu' = d[\mu(E_1 \partial E_2 / E_2 \partial E_1)_{\mu=0}]. \tag{2'}$$

Furthermore, assuming $\mu' = \mu'(E_1)$ (i.e., $\partial\mu'/\partial z = 0$),[1] then from equation (2') it follows that:

$$d\mu' = d\,[\mu(\varkappa_2/\varkappa_1)_{\mu=0}]\,\omega_2/\omega_1,$$

where $\varkappa_2$ and $\varkappa_1$ are the absorption coefficients of the waves E_2 and E_1 at point z. Taking advantage of the fact that at the boundary of the plasma (i.e., at $z = 0$) $\mu' = 0$, and making allowance for the change in the percentage modulation of the interfering wave in the plasma on account of self-interaction (Ref. 20 shows that $\mu = \mu_0[\varkappa_1(z)/\varkappa_1(0)]$), we finally obtain:

$$\mu' = \mu \frac{\omega_2}{\omega_1} \frac{\varkappa_2(z)}{\varkappa_1(z)} - \mu_0 \frac{\omega_2}{\omega_1} \frac{\varkappa_2(0)}{\varkappa_1(0)} = \mu_0 \frac{n_1}{n_2} \frac{\omega_1^2 + \nu_0^2 \tau_0^2}{\tau_0} \left| \frac{\tau_0}{\omega_2^2 + \nu_0^2 \tau_0^2} - \frac{\tau}{\omega_2^2 + \nu_0^2 \tau^2} \right|. \tag{3}$$

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[1] This supposition, as can be seen from formula (3), is justified if the following conditions are fulfilled: that the plasma temperature, the effective electron collision frequency, and also the ratio of the indices of refraction of the waves E_1 and E_2 do not change much in the region of interaction. In the ionosphere the condition of constant temperature can be considered to be well-fulfilled. On the other hand, the collision frequency of an electron can change considerably in the region of interaction. Therefore, computation of the percentage of cross-modulation according to formula (3) for high values of ω_1 and ω_2 (when the condition of constant collision frequency is immaterial), is generally speaking more accurate than for low frequencies. Finally, the condition that the ratio of the indices of refraction be constant is unfulfilled only in the case when the point of reflection of the waves E_1 or E_2 is contained within the region of interaction; the question of the influences of perturbances in the vicinity of the point of reflection E_2 is handled later on in Section 2.

Here it is assumed that $\varkappa_1$, and in a like manner $\varkappa_2$, is defined by the expression:

$$\varkappa_1 = \frac{2\pi e^2 N}{m\,\omega_1 n_1} \cdot \frac{\nu_0 \tau}{\omega_1^2 + \nu_0^2 \tau^2},$$

where e and m are the charge and mass of the electron, N is the electron density, n_1 is the coefficient of refraction of the wave E_1, ν_0 is the effective collision frequency of an electron in the unperturbed plasma (Ref. 7, Section 61), while

$$\tau^2 = \frac{\omega_1^2 + \nu_0^2}{2\nu_0^2}\left[1 + \frac{4\nu_0^2 E^2}{(\omega_1^2 + \nu_0^2) E_n^2}\right]^{1/2} + \frac{\nu_0^2 - \omega_1^2}{2\nu_0^2} = \frac{T_{eff}}{T} \qquad (3')$$

is the quantity which gives the effective temperature, T_{eff}, of the electrons in the plasma under the influence of an unmodulated interfering wave E_1. In equation (3'), E is the field voltage amplitude of the perturbing wave for $\mu_0 = 0$; $E_n = \sqrt{3kTm\delta(\omega_1^2 + \nu_0^2)}/e$ is the characteristic electric field for the plasma (Refs. 19, 20), T is the temperature of the plasma, and k is the Baltzmann constant. The quantity τ, as is clear from equation (3'), is a monotonically increasing function of E, consequently on the boundary of the plasma where $E = E_0$, τ is maximum and equals $\tau(E_0) = \tau_0$; in the interior of the plasma τ monotonically decreases to the value $\tau = 1$ (see Ref. 19).

Equation (3) defines an interesting dependence of the percentage of cross-modulation on the amplitude E_0 of the perturbing wave at the boundary of the plasma, i.e., on the radiated power, P, of the interfering station. A strictly linear growth of μ' as a function of P occurs here only in the case of a weak perturbing field. Actually, as is clear from equation (3), in a strong field and for high frequencies of ω_1 and ω_2 $(\omega_1^2 \gg \nu_0^2\tau_0^2;\ \omega_2^2 \gg \nu_0^2\tau_0^2)$, the percentage of cross-modulation does not exceed the constant value $\mu' = \mu_0 n_1 \omega_1^2 / n_2 \omega_2^2$, no matter what the power of the interfering station. This is also apparent from the curves drawn in Figure 1a (the curves are constructed for various values of K_1, where $K_1 = \frac{\omega_1}{c}\int_{\Delta S} \varkappa_{10}\, ds$ is the total absorption of the perturbing wave in the region

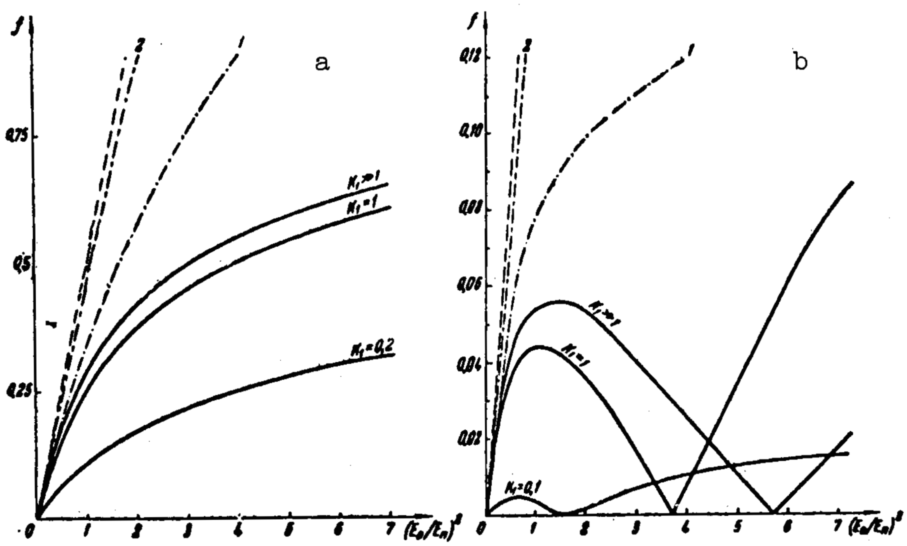

Figure 1. The relationship $f=\frac{\mu'}{\mu_0}\frac{\omega_2^2}{\omega_1^2}\frac{n_1}{n_2}$ for $(E_0/E_n)^2$: a, $\omega_1^2\gg\nu^2$; $\omega_2^2\gg\nu^2$; b, $\omega_1^2=\omega_2^2=2\nu_0^2$.

of interaction of the radio waves; the ratio $(E_0/E_n)^2$, proportional to P, is plotted along the abscissa). In the case when it is not possible to consider the frequency of the perturbing station as high $(\omega_1^2\leqslant\nu_0^2\tau_0^2)$, μ' increases with P at less than a linear rate. However, the nonlinear dependence of μ' and P appears particularly clearly when the frequency of the wave E_2' cannot be considered high: $\nu_0^2\leqslant\omega_2^2\leqslant\nu_0^2\tau_0^2$. In this case, as can be seen from Figure 1b, μ' can decrease as P increases and even reach zero at $\omega_2=\nu_0\sqrt{\tau_0\tau}$. At this point the phase of the cross-modulation changes by π.

In a weak field $(E_0^2\ll E_n^2)$ formula (3) leads to the expression

$$\mu'=\mu_0\frac{e^2E_0^2(\omega_2^2-\nu_0^2)n_1}{6kTm\delta(\omega_2^2+\nu_0^2)^2n_2}\{1-\exp[-2K_1(z)]\}, \tag{4}$$

which agrees with that obtained earlier in Refs. 4, 5. The calculated value of μ', according to formula (4), is plotted with the broken line in Figure 1. It can be seen in the graph that already at $E_0 = E_n$ the use of formula (4) leads to an increased value of μ' (by 1.6-2 times), while for $E_0 = 3E_n$ it is 5-15 times as large. The dot and dash lines in the same Figure plot the dependence of μ' on $(E_0/E_n)^2$ which was obtained in previous works (Refs. 2, 6). The latter did not take into account the self-interaction of the perturbing wave and the necessity of integrating over the whole region of interaction (see Introduction).[1] The curves are constructed so that for a weak field ($E_0^2 \ll E_n^2$), the results of calculating μ' according to the formulas in Refs. 2, 6 and according to formula (3) of the present work agree.[2] In spite of this, however, there is quite a large divergence between them for large field strengths.

During experimental investigation of the dependence of μ' on P, the power of the perturbing station changed in Ref. 8 by up to 100 kw, while in Refs. 5, 11 it reached as much as 520 kw (the distance from the interfering station to the region of interaction is r = 190 km, $\omega_1 = 1.05 \cdot 10^6$ sec^{-1}, $\omega_2 = 4.9 \cdot 10^6$ sec^{-1}, and $\Omega = 2\pi \cdot 400$ sec^{-1} in Refs. 5, 8 and $\Omega = 2\pi \cdot 60$ sec^{-1} in Ref. 11). A comparison of the results of these experiments with the results of computation by formula (3) for the same conditions is shown in Figure 2. It is apparent from the diagram that the experiments described in Refs. 5, 8, 11 are not in contradiction with theory. At the same time, the statement, made in Refs. 4, 5 on the basis of these experiments, that the effective electron

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[1] In constructing the curves in Figure 1 it is assumed that $E = E_0$, where E and E_0 are respectively the amplitudes of the field voltage of the perturbing wave at the "average" point of the interaction region and on the border of the plasma; for curves 2 (the same as in Ref. 6), $E = E_0 \exp\{-1\}$. From Figure 1 it is apparent that the choice of E, which is arbitrary in the old theory, makes itself very strongly felt in the variation of the curve $\mu'(P)$ in the case of a strong perturbing wave.

[2] For this the indeterminate constant $K_2 = \frac{\omega_2}{c}\int \varkappa_2 \, ds$, entering into the formulas of Refs. 2, 6, is appropriately chosen.

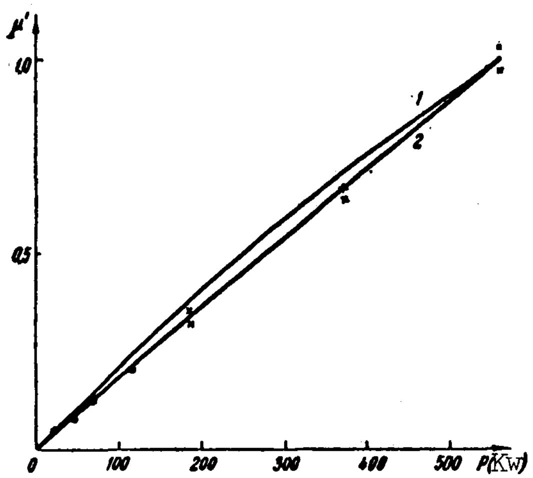

Figure 2. The dependence of μ'/μ'_{max} on P for the conditions in Ref. 11 (curve 1) and in Refs. 5, 8 (curve 2); o, x are respectively the results of experiments in Refs. 8, 11

temperature changes less than 10 percent under the influence of a 520 kw perturbing wave is unfounded, because in the case under examination, evidently, it is not possible to determine the deviation of the function $\mu'(P)$ from the linear, even if the effective temperature changes by 50 percent (see Figure 2).[1]

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[1]Evaluating E_n for these conditions we have: $E_n \approx 27.5$ mv·m^{-1}, whereas for P = 520 kw: $E_0 = 36 \cos \beta \approx 20$ mv·m^{-1}. Here consideration is taken of the fact that because of the influence of the earth's magnetic field for $\omega_1 \ll \omega_H$ the total vector of the electric field intensity does not act in causing the perturbation in the plasma, but instead only in its projection along the direction of the magnetic field $E \cos \beta$; the average value of $\cos \beta$ is about 0.58. Hence, the perturbing wave at maximum-radiated power P = 520 kw caused a change in the effective electron temperature on the average of about 50 percent, i.e., it was not sufficiently strong. This latter circumstance was due mostly to the fact that the region of interaction was located at a substantial distance from the interfering station ($r \approx 190$ km); if the interaction had taken place directly above the perturbing station ($r \sim 100$ km), it would have been

Formula (3) was obtained above under the condition that μ_0 is small. Analysis shows, however, that it can be used to calculate μ' with good accuracy (to 15 percent) for any value of μ_0.

It is also necessary to note that cross-modulation of radio waves in plasma was examined only for low modulation frequencies $\Omega \ll \delta\nu$. In order to solve the problem for an arbitrary value of Ω, it is first necessary to find the nature of the perturbations caused by the interfering wave to the effective electron temperature in the general case (see Ref. 20), and then determine how the absorption of the wave E_2 changes under the influence of these perturbations. The appropriate operations lead to the simple result for the case of small $\mu_0 (\mu_0 \lesssim 1/2)$, and for high frequency of the perturbing wave $(\omega_1^2 \gg \nu_0^2)$:

$$\mu'(\Omega) = \mu'(0) \frac{\delta\nu_0\tau_0}{\sqrt{\Omega^2 + (\delta\nu_0\tau_0)^2}}; \quad \operatorname{tg}\varphi' = \Omega/\delta\nu_0\tau_0, \tag{5}$$

where $\mu'(0)$ is given by expression (3). From equation (5) it is apparent that the dependence of the percentage of cross-modulation and the phase of cross-modulation on Ω, obtained in Ref. 1, preserves its general appearance in the case being examined; it is only necessary to exchange the ν of Ref. 1 (an "average" collision frequency in the region of interaction) for $\nu_0\tau_0$.

(footnote continued from previous page)
possible to expect a more substantial nonlinearity in the dependence of μ' on P (see Figure 1a). However, the nonlinearity appears most clearly for low values of ω_2 (see Figure 1b). In order to observe most clearly the nonlinearity, it is necessary to make observations for a sufficiently large number of values of P, some measurements being made for small values of $P \sim 50$-100 kw (in order to find the path of the straight line showing linear dependence of μ' on P). We also note that in Refs. 5, 8 the frequency of modulation was poorly chosen (for high modulation frequencies the dependence of μ' on P is more closely linear) (see equation (5)).

2. Resonance of the Cross-Modulation near the Gyrofrequency

In examining the question of resonant increase of the percentage of cross-modulation at frequencies ω_1 of the perturbing wave which are very close to the gyrofrequency ω_H, it suffices to consider only the influence of an extraordinary perturbing wave,[1] insofar as an ordinary wave causes only a negligible cross-modulation ($\mu' \leqslant 1$ percent for $P \leqslant 500$ kw), and besides, the perturbations caused by this wave in the plasma do not possess resonant properties at $\omega_1 \sim \omega_H$. Then the percentage of cross-modulation as before is determined by formula (3), where it is necessary to replace ω_1^2 by $\omega_1'^2 = (\omega_1 - \omega_H)^2$ and E_0 by $E_0' = E_0/\sqrt{2}$ (see Refs.

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[1]Here for simplicity it has been assumed that the interfering wave is propagated longitudinally. We note in this connection that, as Zheleznyakov has shown in Ref. 22, in the case of nonlongitudinal propagation of the interfering wave the dependence of μ on ω can, generally speaking, change in consequence of the influence of polarization corrections. Specifically, resonance effects no longer occur at the gyrofrequency ($\omega_{res} = \omega_H$), but at another frequency, the magnitude of which changes from ω_H to $\omega_H \sqrt{1 + \omega_0^2/\omega_H^2}$ depending on the direction of propagation of the wave (here $\omega_0 = \sqrt{4\pi N e^2/m}$ is the Langmuir frequency of the plasma). However, the examination being carried out in the present section shows that resonance of the cross-modulation is accomplished only under the condition that the interaction of the waves takes place at the beginning of the ionospheric layer in the region where the Langmuir frequency is not large ($\omega_0 \lesssim 2 \cdot 10^6$). The role of the polarization correction under these conditions is negligible: for example the shift of the resonance frequency is $\frac{\omega_{res} - \omega_H}{\omega_H} \leqslant \frac{1}{2}\frac{\omega_0^2}{\omega_H^2} \leqslant 2\%$. Due to this, the resonance effect and the shape of the resonance curve in the first approximation do not depend on the direction of propagation of the perturbing wave, hence it suffices to limit ourselves to the simple case of longitudinal propagation.

20, 21). However, as has been noted above, the conditions for applying formula (3) are not fulfilled if the point of reflection of the wave E_2 lies in the region of interaction. At the same time, in the experiments (Refs. 15, 16) in which resonance was observed, the vicinity right at the point of reflection of the wave E_2 was subjected to perturbation; this circumstance is emphasized in Ref. 16. Therefore, in examining the topic of cross-modulation resonance near the gyrofrequency it is necessary to make allowance for the influence of the interfering wave on the point of reflection of the wave E_2.

Assume for simplicity that the interfering wave is weak and that ω_2 (the frequency of the wave E_2) is larger than the collision frequency, ν_0, of an electron. In this case, the percentage of cross-modulation is determined by the following expression (Ref. 7, Section 64):

$$\mu' = \frac{\omega_2}{c}\int\limits_S \varkappa_2 \frac{\Delta\nu}{\nu_0}\, ds = \frac{\omega_2}{c}\int\limits_S \varkappa_2 \mu_0 \frac{e^2E^2(s)}{3mkT\,\delta\,(\omega_1'^2+\nu_0^2)}\, ds, \qquad (6)$$

where $\Delta\nu$ is the amplitude of the periodic perturbations of the collision frequency of an electron which are caused by the interfering wave, $\varkappa_2$ is the coefficient of absorption of the wave E_2, $\omega_1' = \omega_1 - \omega_H$, and E(s) is the amplitude of the perturbing wave at point s; the integration is carried out over the whole path of the wave E_2 in the plasma S. We now take into account the fact that for $\omega_2^2 \gg \nu_0^2$ the total absorption of the wave E_2 in the ionosphere (including the point of reflection) is almost always given by the formula:

$$K_2 = \frac{\omega_2}{c}\int\limits_S \varkappa_2\, ds = \frac{2\omega_2}{c}\int\limits_0^{z_0} \frac{2\pi\sigma_2}{\omega_2\sqrt{\varepsilon_2}}\, dz\,[1 + 0(\nu^2/\omega_2^2)], \qquad (7)$$

where z_0 is the point of reflection of the wave E_2, σ_2 and ε_2 are the conductivity and dielectric permittivity for this wave.[1] Then making a trivial transformation in equation (7),

$$\frac{2\pi\sigma_2}{\omega_2\sqrt{\varepsilon_2}} = \frac{2\pi\sigma_2}{\omega_2\cos\varphi_2} + \frac{2\pi\sigma_2(\cos\varphi_2 - \sqrt{\varepsilon_2})}{\omega_2\sqrt{\varepsilon_2}\cos\varphi_2}$$

(φ_2 is the angle of incidence of the wave E_2), substituting equation (7) in the expression for μ' and remembering that

$$\frac{\omega_2}{c}\frac{2\pi\sigma_2}{\omega_2} = \frac{\omega_1}{c}\frac{2\pi\sigma_1}{\omega_1}\frac{\omega_1'^2+\nu_0^2}{\omega_2^2+\nu_0^2} = -\frac{dE}{dz}\frac{\omega_1'^2+\nu_0^2}{\omega_2^2+\nu_0^2},$$

it is easy to carry out the integration in equation (6) and obtain the following expression for the percentage of cross-modulation:

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[1]In fact, taking advantage of the known exact solution for the linear layer (see Ref. 7, Section 66), it is easy to show that formula (7) is correct for the linear layer:

$$K \cong -\operatorname{Im}\left(\frac{4}{3}\frac{\omega}{c}z_0\frac{(1+i\alpha)^{3/2}}{1+i\beta}\right) \approx$$

$$\approx \frac{2\omega}{c}z_0\left(\alpha+\frac{2}{3}\beta\right)\left\{1+0(\beta^2,\alpha^2,\alpha\beta)\right\} = \frac{2\omega}{c}\int_0^{z_0}\frac{2\pi\sigma}{\omega\sqrt{\varepsilon}}dz\left\{1+0(\nu_0^2/\omega^2)\right\}$$

(here $\alpha + \beta z/z_0 = 4\pi\sigma/\omega$). Joining now, as is common, the solution for the linear part with the approximation of geometric optics for the remaining part of the layer and taking into account the fact that in the region where geometric optics is applicable the coefficient of absorption $\varkappa$ equals $2\pi\sigma/\omega\sqrt{\varepsilon}$ with accuracy $0(\gamma\nu_0^2/\omega^2)$, we arrive at formula (7). We note that $\gamma = z_0(\partial^2\varepsilon/\partial z^2)_{z=z_0}/4(\partial\varepsilon/\partial z)_{z=z_0}$ and, consequently, formula (7) is inapplicable only for $(\partial\varepsilon/\partial z)_{z=z_0}\approx 0$, i.e., for frequencies close to the critical frequency of the layer.

$$\mu' = \mu_0 \frac{e^2 E_0^2}{6kTm\,\delta\,\omega_2^2 \cos\varphi_2} [1 - \exp\{-2K_1^0\} + f(K_1^0)] =$$
$$= \mu'_\infty [1 - \exp\{-2K_1^0\} + f(K_1^0)]. \tag{8}$$

Here $K_1^0 = \frac{\omega_1}{c} \int_0^{z_0} \varkappa_1 dz$ is the absorption of the disturbing wave up to the point of reflection of the wave E_2, and

$$f(K_1^0) = \frac{2\omega_1}{c} \int_0^{z_0} \varkappa_1 \exp\{-2K_1^0\} \frac{\cos\varphi_2 - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_2}} dz. \tag{8'}$$

If in equation (8') we set $\varepsilon_2 = \varepsilon_2(0) = \cos^2\varphi_2$, then $f(K_1^0) \equiv 0$ and formula (8) coincides with formula (3) (for $n_2 = \cos\varphi_2$, $n_1 = 1$), as it must be.[1] Consequently, $\Delta\mu' = \mu'_\infty f(K_1^0)$ is the change in μ', which arises on account of the deviation of $\epsilon_2(z)$ from its value at the boundary of the plasma ($\epsilon_2(0) = \cos^2\varphi_2$); the largest contribution to $\Delta\mu'$, naturally, is given by the region surrounding the point of reflection of the wave E_2, at which $\epsilon_2 \sim 0$. The form of the function $f(K_1^0)$ is only slightly dependent on the character of the change of the electron density $N(z)$ with the height z. This can be seen from Figure 3 which shows graphs of

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[1]It is necessary to note that generally speaking the factor 1/2 must appear in formula (8) on account of the fact that the effective power of an extraordinary perturbing wave is only half as large as the power of a plane polarized wave ($E_0' = E_0/\sqrt{2}$). However, in formula (8) it is assumed that the whole path of the wave in the plasma is subject to interference, whereas above (in Section 1) the cross-modulation was examined only for one direction of the path of the wave E_2 (for example, up to the point of reflection); this is what leads to the inverse factor 2.

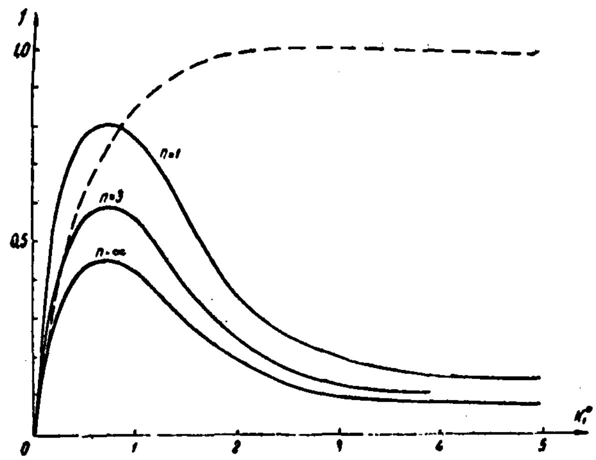

Figure 3

the function $f(K_1^0)$ for various types of dependence of N on z: $N = N_0(z/z_0)^n$, where $n = 1, 3, \infty$. The broken line in the same figure plots the curve $1 - \exp\{-2K_1^0\}$; comparing it with the curve $f(K_1^0)$, we see that for small values of K_1^0 a consideration of the correction $\Delta\mu'$ is quite important, while for large K_1^0 the correction is negligible, as it must be; the maximum value of $\Delta\mu'$ is 1.5-2 times smaller than μ'_∞.

Now let us examine expression (8). First, it follows from it that the percentage of cross-modulation depends only slightly on the frequency of the interfering wave in the cases when the interfering wave attenuates sufficiently in the region of interaction (i.e., for $K_1^0 \gtrsim 1$):

$$\mu' \approx \mu'_\infty = \mu_0 \frac{e^2 E_{||}^2}{6kTm\delta\omega_2^2 \cos\varphi_2}. \tag{9}$$

From this it is clear that the percentage of cross-modulation at the gyrofrequency (for $\omega_1 \approx \omega_H$) does not necessarily exceed the percentage of cross-modulation at some other frequency ω_1 for which $K_1^0 \gtrsim 1$. The reason for this effect is that, although the extraordinary wave E_1 for

$\omega_1 \approx \omega_H$ does cause very strong perturbations in the plasma, it damps out in a very thin layer. Conversely, if the frequency E_1 is substantially different from the gyrofrequency, then, although the perturbations it causes are substantially weaker than for $\omega_1 \approx \omega_H$, the disturbed layer is, correspondingly, substantially thicker. Therefore, the total percentage of cross-modulation does not depend on the frequency of the perturbing wave, if the wave E_2 passes through the whole perturbed layer ($K_1^0 \gtrsim 1$); in this case "full", as it were, cross-modulation takes place.

This conclusion (as was also made in Ref. 5) is in agreement with the results of experiments carried out in England (Refs. 5, 8, 11). In these experiments, the cross-modulation for $\omega_1 \approx \omega_H$, $P \sim 100$ kw, $\omega_2 = 4.9 \cdot 10^6$ sec^{-1} was not stronger than that usually observed for $\omega_1 = 10^6$ sec^{-1}, $\omega_2 = 4.9 \cdot 10^6$ sec^{-1} and $P \sim 100$ kw (see Introduction).

However, from formula (8) it also follows that when the frequency ω_2 decreases, the maximum value of the percentage of cross-modulation increases proportionally to $1/\omega_2^2$:[1]

$$\frac{\mu'\max}{\mu_0}\frac{\cos\varphi_2}{P_{(\kappa в)}} \approx (1.6-1)\frac{e^2(E_0^2/P)}{6kTm\delta\omega_2^2} \approx (8-5)\,10^{-3}\left(\frac{10^6}{\omega_2}\right)^2. \tag{9'}$$

This relationship, as can be seen in Figure 4, is supported by the data of all experiments on cross-modulation at the gyrofrequency.

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[1]Here it is important that in the derivation of formula (8) the influence of the earth's magnetic field on the wave E_2 is neglected, which is justified only for an ordinary wave undergoing transverse propagation. In equation (9') it is assumed that the distance from the interfering station to the region of interaction is equal to 100 km; moreover, the fact that the perturbing wave does not always propagate longitudinally is taken into account (this leads on the average to a decrease of the effective power by 1.5 times).

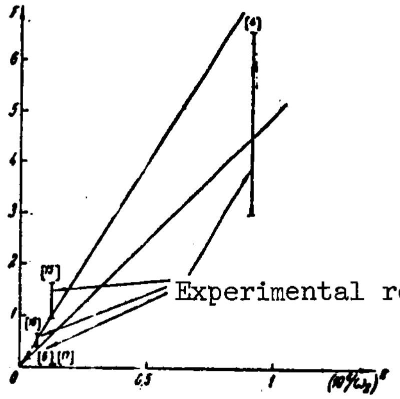

Figure 4. $F = \frac{\mu'_{max}}{\mu_0} \frac{\cos\varphi_2}{P} 10^3$ as a function of $(10^6/\omega_2)^2$

In analyzing the variation of the dependence of μ' on ω_1 for low values of ω_2, we become convinced that, in that case when ω_1 is close to ω_H, a well defined resonant increase in the percentage of cross-modulation takes place. The reason for the resonance is that the wave E_2 for low frequencies of ω_2 (or for a large angle of incidence φ_2) propagates only within a very thin layer of plasma, due to which "full" cross-modulation ($K_1^0 \gg 1$) takes place only when ω_1 is close to ω_H. The variation in μ' as a function of ω_1, for $\omega_1 \sim \omega_H$ and for low frequencies of the wave E_2 is plotted in Figure 5a (solid curves). It is apparent from the diagram that the resonance curve has either one hump or two humps; two humps are clearly evident in the cases when $K_1^0(\omega_H) > 1$. The broken line in the same figure shows the curve constructed without taking into account the influence of the perturbing wave on the point of reflection of the wave E_2; from the figure it is evident that taking $\Delta\mu'$ into account substantially influences the shape of the resonance curve. The shape of the resonance curves drawn in Figure 5a agrees very well with experiment.

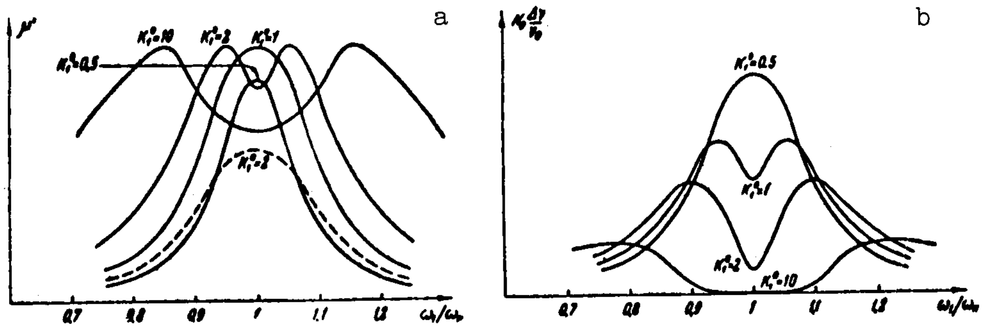

Figure 5

Figure 5b shows the known curves (Ref. 2) which depict $N_0 \frac{\Delta \nu}{\nu_0}$ as a function of $\frac{\omega_1}{\omega_H}$. They are constructed for the very same cases as the curves in Figure 5a. Comparing both figures, we become convinced that the variation of μ' and $N_0 \frac{\Delta \nu}{\nu_0}$ as functions of the frequency ω_0 differs substantially.[1]

3. The Phase of Cross-Modulation

In Ref. 3 it is shown that the result of calculating the cross-modulation with the help of kinetic theory, taking into account the electron velocity distribution in the plasma, agrees with an accuracy of not less than 10-15 percent with the result of the usually accepted "elementary" theory (Ref. 1). However, an expression for the electron velocity distribution, and consequently also for the percentage of cross-modulation, was found only for the extreme instances of high $(\Omega \gg \delta\nu)$ or low $(\Omega \ll \delta\nu)$ modulation frequencies. Specifically, this made it impossible to determine the phase, φ', of the cross-modulation as a function of the modulation frequency. At the same time, measurements

[1] In Ref. 2, it was assumed that the depth of the cross-modulation is proportional to $N_0 \frac{\Delta \nu}{\nu_0}$, where N_0 is the electron density at the point of reflection of the wave E_2.

of φ' as a function of Ω in cross-modulation are the most accurate (Refs. 8, 10) and, therefore, it is important to know to what result kinetic theory leads in this case (elementary theory leads, as is well-known, to the simple relation tg $\varphi_{elem} = \Omega/\delta\nu$).

The result of the appropriate calculation is developed in the present section. For simplicity, the interfering wave is assumed to be weak. The following approximations for the distribution function are found by the usual method in the power series expansion of $\delta\nu_0/\Omega$ (for $\Omega \gg \delta\nu_0$) and of $\Omega/\delta\nu_0$ (for $\Omega \ll \delta\nu_0$). These have the following appearance ($\omega_1^2 \gg \nu_0^2$):

$$f_0 = f_{00} + \frac{2\mu_0 e^2 E_0^2}{3kTm\delta\omega_1^2} f_{00}\left[\left(u^2 - \frac{3}{2}\right)\cos\Omega t + \right.$$
$$\left. + \frac{16}{3\sqrt{\pi}}\frac{\Omega}{\delta\nu_0}\left(u - \frac{2}{\sqrt{\pi}}\right)\sin\Omega t + O\left(\frac{\Omega^2}{\delta^2\nu_0^2}\right)\right]$$

if $\Omega \ll \delta\nu_0$, and

$$f_0 = f_{00} + \frac{\sqrt{\pi}\, e^2 E_0^2 \nu_0\mu_0}{32kTm\Omega\omega_1^2} f_{00}\left\{\left[(4u^3 - 8u) + \frac{9\pi}{64}\left(\frac{\delta\nu_0}{\Omega}\right)^2(-12u^5 + 55u^3 - \right.\right.$$
$$\left. -38u)\right]\sin\Omega t + \left[\frac{3\sqrt{\pi}}{8}\frac{\delta\nu_0}{\Omega}(6 - 16u^2 + 6u^4) - \frac{27\pi^{3/2}}{256}\left(\frac{\delta\nu_0}{\Omega}\right)^3\left(\frac{57}{2} - \right.\right.$$
$$\left.\left.\left. - \frac{905}{4}u^2 + \frac{375}{2}u^4 - 30u^6\right)\right]\cos\Omega t + O\left(\frac{\delta^4\nu_0^4}{\Omega^4}\right)\right\},$$

if $\Omega \gg \delta\nu_0$. Here $u = v/\sqrt{2kT/m}$, and $f_{00}(v)$ is the Maxwell distribution function for the electron velocities.

Calculating the phase of the cross-modulation, we find that for $\Omega \ll \delta\nu_0$:

$$\mathrm{tg}\,\varphi_k' = \left(8 - \frac{64}{3\pi}\right)\frac{\Omega}{\delta\nu_0} \approx 1{,}22\,\frac{\Omega}{\delta\nu_0},$$

while for $\Omega \gg \delta\nu_0$:

$$\operatorname{tg}\varphi_k' = \frac{\Omega}{\delta\nu_0}\left[1 + \frac{9\pi}{128}\left(\frac{\delta\nu_0}{\Omega}\right)^2\right] \approx \frac{\Omega}{\delta\nu_0}\left[1 + 0.22\left(\frac{\delta\nu_0}{\Omega}\right)^2\right].$$

The expressions obtained make it possible to determine (approximately) $\operatorname{tg}\varphi_k'/\operatorname{tg}\varphi'_{elem}$ as a function of $\Omega/\delta\nu_0$. This is shown in Figure 6.

As can be seen from the figure, the deviation from the result of elementary theory is rather great; apparently, it can be experimentally checked. The appropriate measurements would permit one to establish the character of the electron velocities distribution in the ionosphere (in the present computation, it has been assumed to be Maxwellian).

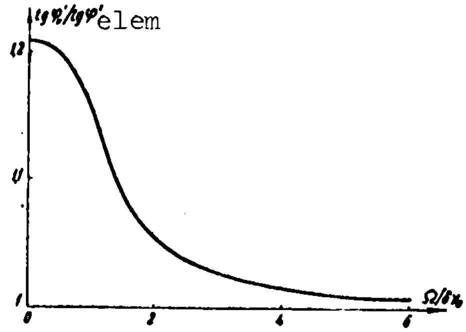

Figure 6

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